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CHARLES F. HAGER

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Charles F. Hager

APPLICATIONS OF THE WEIBULL DISTRIBUTION

by

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Lieutenant, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE

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APPLICATIONS OF THE WEIBULL DISTRIBUTION

by

Charles F. Hager

This work is accepted as fulfilling
the thesis requirements for the degree of
MASTER OF SCIENCE

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United States Naval Postgraduate School

ABSTRACT

A summary of the Weibull distribution and three representative sampling procedures to determine point and confidence interval estimates of the parameters that occur in the functional form of the Weibull distribution are presented. Following this, a model, assuming the Weibull distribution, is proposed which could have possible applications in analyzing Polaris Missile System Trouble Failure Reports to determine point estimates of the reliability of the Polaris Missile components.

I am indebted to Professor P. W. Zehna for his continued patience, encouragement, and most capable guidance while acting as faculty advisor. I also wish to thank Professor W. M. Woods for his valuable assistance as second reader.

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LIST OF ILLUSTRATIONS

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TABLE OF SYMBOLS

(Listed in order of their use in the text)

X	a random variable with the three parameter Weibull distribution
$F(x)$	the cumulative distribution function for X
$f(x)$	the density function for X
λ	the mean of a random variable
σ^2	the variance of a random variable
$\Gamma(\cdot)$	the gamma function
T	operating time to failure
$f(t)$	the density function for T
$F(t)$	the cumulative distribution function for T
$R(t)$	the reliability function
$\phi(u)$	the characteristic function
$Z(t)$	the hazard function or instantaneous failure rate
$G(t)$	the cumulative distribution function for a complex item
$g(t)$	the density function for a complex item
L	the likelihood function

CHAPTER I

INTRODUCTION

This thesis is concerned with summarizing the Weibull distribution and providing several possible methods of analyzing Polaris Missile System Trouble Failure Reports to determine the reliability of Polaris Missile components when the Weibull distribution is assumed to characterize the failure times of the components.

The Weibull distribution, which was first proposed by Waloddi Weibull [1] in 1951, is a member of a class of distributions which characterize wear out failure; the longer an item has been used the greater the probability of failure. Wear out failure is associated with gradual depletion of material, fatigue, or accumulated shocks and other factors.

The key assumption underlying many of the statistical procedures in current use for evaluating the reliability of components or equipment is that the failure times follow the exponential distribution. However, in many practical situations sufficient data is not usually available to justify the assumption that the exponential failure law truly characterizes the time to failure distribution of the equipment or components being tested.

The Weibull distribution which includes the exponential distribution as a special case is a more general assumption about the time to failure law of components or equipment. Kao [2] has shown that the failure age distribution of electron tubes is better chacterized by the Weibull distribution with shape parameter approximately equal to 1.7 than by the exponential distribution. Also when the exponential distribution is assumed to be the time to failure law, M. Zelen and M. C. Dannemiller [3] demonstrate that four commonly employed sampling procedures are not robust* with respect to Weibull alternatives.

Chapter II summarizes the distribution theory associated with the Weibull distribution and defines terms to be used in the remainder of this thesis. Chapter III outlines sampling plans which can be employed to obtain estimates of the parameters from experimental measurements. The various estimates to follow are derived under each of these plans. Chapter IV proposes three possible methods of using Polaris Missile Trouble Failure Reports for determining the reliability of Polaris system components.

* Sampling procedures which are not very sensitive to departures from basic assumptions are called robust.

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CHAPTER II

SUMMARY OF THE WEIBULL DISTRIBUTION

Let X be a random variable which may take on any value in the continuous scale from m to infinity where m is a non-negative number. The three parameter Weibull cumulative distribution function (c.d.f.) is defined as

$$(1).\quad F(x) = \begin{cases} 1-e^{\frac{-(x-m)^b}{a}} & \text{for } x \geq m, \text{ where} \\ 0 & x < m \end{cases}$$

a = scale parameter, $a > 0$

b = shape parameter, $b > 0$

m = location parameter*, $m \geq 0$

The probability density function is defined as the first derivative of F(x) with respect to x, and is given by,

$$(2).\quad \frac{dF(x)}{dx} = f(x) = \begin{cases} \frac{b(x-m)}{a}^{b-1} e^{\frac{-(x-m)^b}{a}} & , x \geq m \\ 0 & x < m \end{cases}$$

Weibull [1] demonstrated that yield strength of Bofors steel, size distribution of fly ash, and fiber

* x_0, m, and x_m, respectively were the notations used by Weibull [1] .

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strength of Indian cotton are distributed in good agreement with the cumulative distribution function, equation (1). Kao [4] has shown that the cumulative distribution function, equation (1), qualifies as a failure distribution and has successfully employed the Weibull distribution with location parameter, m, equal to zero to characterize the time to failure of electron tubes. For the purposes of this thesis and the sampling plans which follow in Chapter III, the location parameter, m, will be assumed to be zero. This is a reasonable assumption since if an item is placed in operation at time zero, it is exposed to the risk of failure from the time it is first put into use.

For completeness of presentation, the moments and characteristic function of the three parameter Weibull distribution are listed below, but a detailed discussion will only be presented for the two parameter sub-family having $m = 0$. Given $f(x)$ as the above probability density function, the nth moment is defined by the formula

$$E(X^n) = \int_m^{\infty} x^n \, \frac{b(x-m)^{b-1}}{a} \, e^{\frac{-(x-m)^b}{a}} \, dx$$

and is given by

$$E(X^n) = \sum_{K=0}^{n} \binom{n}{K} a^{\frac{k}{b}} m^{n-k} \, \Gamma\left(\frac{k}{b} + 1\right)$$

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The mean and variance, denoted by λ and σ^2 respectively, are easily determined from the above relation to be

$$\lambda = m + a^{\frac{1}{b}} \Gamma\left(\frac{1}{b} + 1\right)$$

and

$$\sigma^2 = a^{\frac{2}{b}} \left[\Gamma\left(\frac{2}{b} + 1\right) - \Gamma^2\left(\frac{1}{b} + 1\right) \right]$$

respectively. The function $\Gamma(\cdot)$ is the gamma function which is defined for every $v > 0$ by

$$\Gamma(v) = \int_0^\infty x^{v-1} \, e^{-x} \, dx$$

Suppose now that T is a random variable which may take on any value in the continuous time scale from zero to infinity, and the random variable T represents the total time to failure of an item. Let us further assume that the two parameter Weibull probability density function, f(t), specifies the probability law of the random variable T where

$$(3). \quad f(t) = (bt^{b-1}/a) \exp(-t^b/a), \quad t \geq 0; \; a, b, > 0$$

$$= 0 \qquad \text{elsewhere}$$

Accordingly, if an item is placed in operation at time zero, the probability that the total life of the item is less than or equal to t is expressed as

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$$P(T \leq t) = \int_0^t (bt^{b-1}/a) \exp(-t^b/a)dt = 1-\exp(-t^b/a)$$

Therefore the cumulative distribution function of the random variable T is given by

(4). $F(t) = 1 - \exp(-t^b/a)$

The reliability function, denoted by R(t), of an item is also of particular interest since it is desirable to know the probability that an item or component does not fail in the time interval (0,t). Thus the reliability function is the probability that the time to failure of an item is at least t time units and is expressed by

$$R(t) = P(T > t) = \int_t^\infty (bt^{b-1}/a)\exp(-t^b/a)dt, \text{ for } t > 0$$

which upon integration becomes

(5). $R(t) = \exp(-t^b/a) \; ; \; t > 0$

Note that the reliability function is merely the unity complement of the cumulative distribution function, equation (4).

The nth moment of the random variable T is defined by the formula

$$E(T^n) = \int_0^\infty t^n (bt^{b-1}/a) \exp(-t^b/a)\, dt$$

and is given by

$$(6).\quad E(T^n) = a^{\frac{n}{b}} \Gamma\left(\frac{n}{b} + 1\right)$$

The characteristic function of the random variable T is

$$\phi(u) = E(e^{iuT}) = \int_0^\infty e^{iut} \frac{b}{a} t^{b-1} e^{-(t^b/a)}\, dt$$

which upon integration becomes

$$\phi(u) = \sum_{j=0}^{\infty} \frac{(iu)^j}{j!} a^{j/b} \Gamma\left(\frac{j}{b} + 1\right)$$

The mean time to failure of the random variable T, denoted by λ, is defined as the first moment about zero and is easily determined from equation (6) to be

$$(7).\quad \lambda = a^{\frac{1}{b}} \Gamma\left(\frac{1}{b} + 1\right)$$

In a similar manner, the variance denoted by σ^2 of the random variable T is determined by the formula,

$E(T^2) - E(T)^2$, to be

$$\sigma^2 = a^{\frac{2}{b}} \left[\Gamma\left(\frac{2}{b} + 1\right) - \Gamma^2\left(\frac{1}{b} + 1\right) \right]$$

Another function of a time to failure probability distribution which provides a great deal of insight into the probability law is the hazard function or instantaneous failure rate which will be denoted by Z(t). The instantaneous failure rate or hazard function is defined by

$$Z(t) = \frac{f(t)}{R(t)}$$

where f(t) and R(t) are the density and reliability functions respectively. Substitution of f(t) and R(t) of equations (3) and (5) in the above expression,we have

$$(8). \quad Z(t) = bt^{b-1}/a \quad \text{for } t > 0$$

It is interesting to note that, for values of the shape parameter greater than one, the instantaneous failure rate is an increasing function of time, therefore indicating that wear out occurs during the life of the item. But in the special case where the shape parameter equals one and the Weibull distribution simplifies to the exponential distribution, it is observed that the instan-

taneous failure rate is a constant equal to 1/a, independent of age.

The effect of varying the shape and scale parameters can most readily be illustrated by graphs of the probability density function because the probability that the random variable T will lie in the interval (t, t+h) can be interpreted geometrically as the area under the probability density function in the interval from t to t+h. Figure one illustrates the probability density function for several values of the shape parameter, b, and with the scale parameter, a, equal to one. The effect of changing the value of the scale parameter is to merely squeeze or broaden the graph of the probability density function, i.e., an increase in the value of the scale parameter would broaden while a decrease in the value of the scale parameter would squeeze the graph of the probability density function together. The first derivative of f(t) with respect to t is

$$f'(t) = \frac{b}{a} t^{b-2} e^{\frac{-t^b}{a}} \left[\frac{-b}{a} t^b + (b-1) \right]$$

which shows that f(t) has a maximum at

$$t = \left[\frac{(b-1)a}{b} \right]^{\frac{1}{b}}$$

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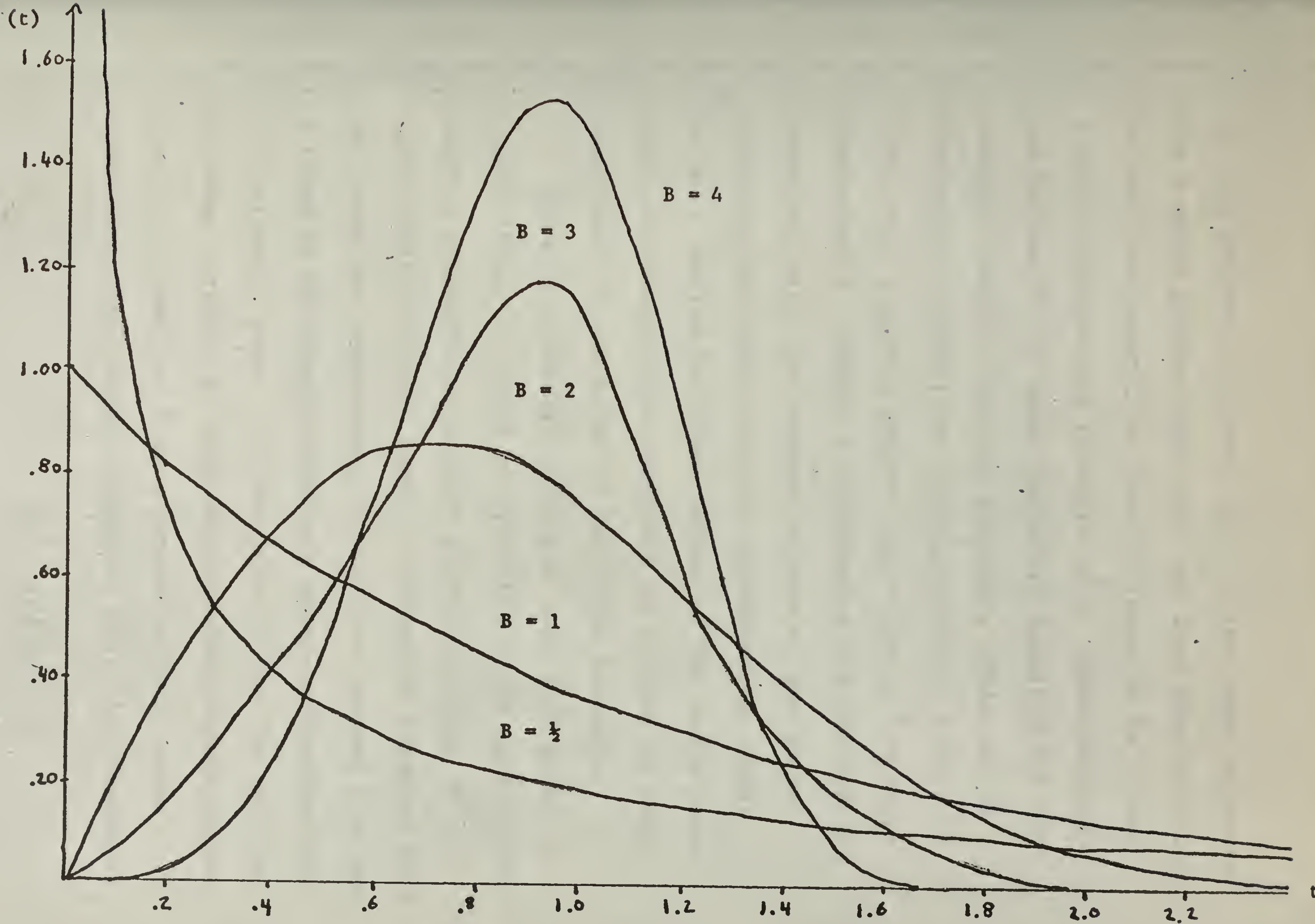

Figure 1. Graph of two parameter Weibull probability density function for several values of shape parameter, b, and scale parameter, a, equal to one.

if b is greater than one. But if the shape parameter has a value greater than zero but less than or equal to one, the probability density function is a monotonically decreasing function of t. Thus increasing values of the shape parameter squeezes the probability density function together. This can be interpreted intuitively to mean that the faster an item wears out, the greater the value of the shape parameter, b. Figure 2 illustrates the reliability function, R(t), for several values of the shape parameter and with the scale parameter, a, equal to one. Notice for fixed values of the scale parameter, all the curves intersect at t equal to one, regardless of the value of the shape parameter.

Thus far, consideration has only been given to a single item. It would be desirable to be able to consider a component or complex item consisting of M individuals or single items. Instead of assuming that the items are independent, it will be assumed that the M items of the complex item are quasi-independent in the following sense. If the life of the ith item is a random variable T_i with corresponding density function $F_i(t)$ then let T be a random variable whose cumulative distribution function, G(t), is determined

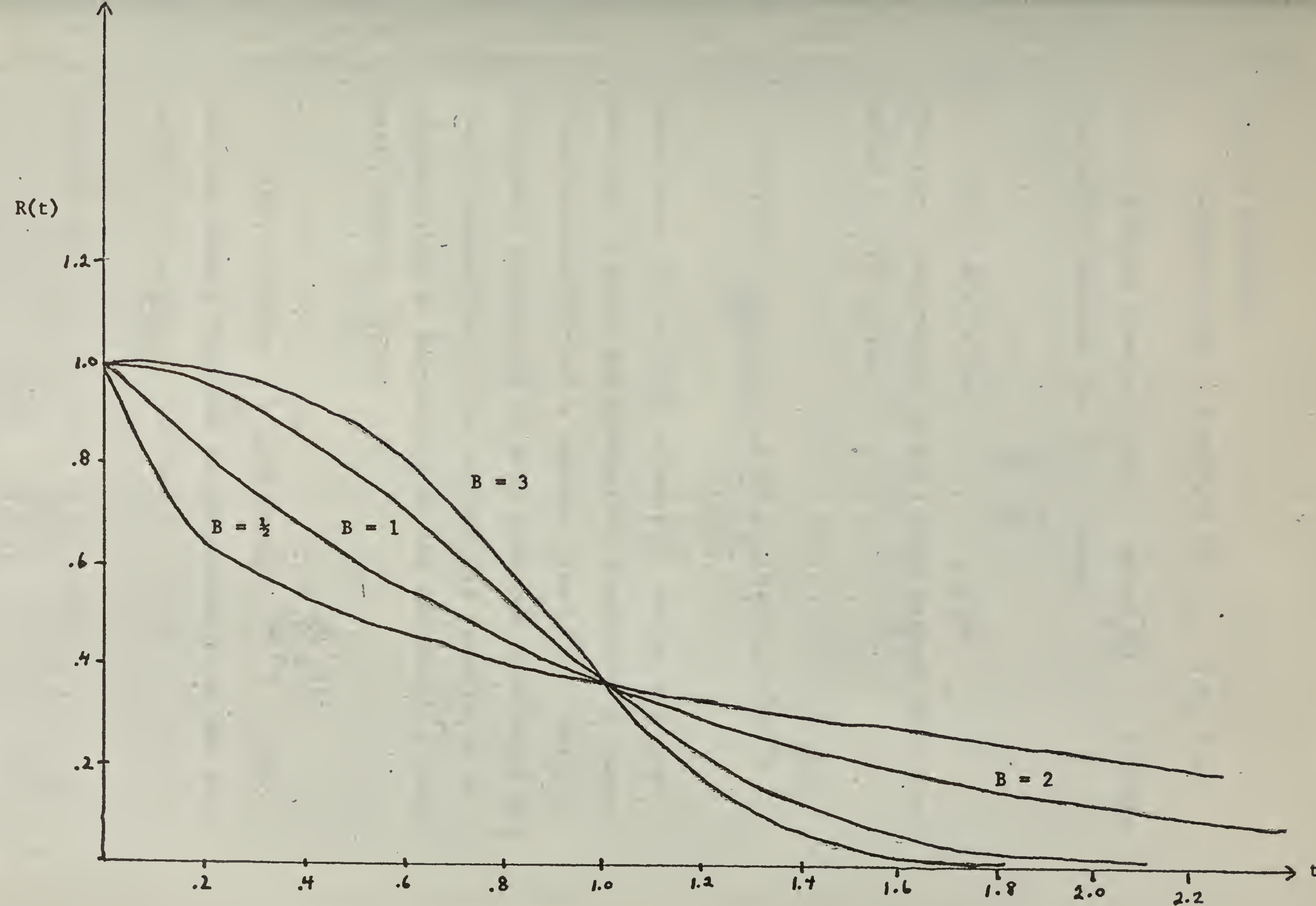

Figure 2. Graph of reliability function for several values of shape parameter, b, and scale parameter, a, equal to one.

by the equation

$$1-G(t) = P(T_1 > t)P(T_2 > t) \ldots P(T_M > t)$$

for t greater than or equal to zero [5] . Solving the above equation for G(t) yields

$$(9). \quad G(t) = 1-e^{-\sum_{i=1}^{M} \frac{t^{b_i}}{a_i}}, \quad t \geq 0$$

The first derivative of G(t) with respect to t denoted by g(t) gives

$$(10). \quad \frac{dG(t)}{dt} = g(t) = \sum_{i=1}^{M} \frac{b_i t^{b_i - 1}}{a_i} e^{-\sum_{i=1}^{M} \frac{t^{b_i}}{a_i}}$$

Equations (9) and (10) above will define the probability density and cumulative distribution function respectively of a complex item [5] , i.e., a component consisting of more than one item which are quasi-independent. The reliability function of a complex item is determined by the probability statement, $P(T > t)$, to be

$$(11). \quad R(t) = \int_t^{\infty} g(t)\, dt = e^{-\sum_{i=1}^{M} \frac{t^{b_i}}{a_i}}, \quad t > 0$$

and the hazard function is determined from the relation $Z(t) = \frac{g(t)}{1-G(t)}$ to be

$$(12).\quad Z(t) = \sum_{i=1}^{M} b_i t^{b_i - 1}/a_i$$

The formula for computing the nth moment of a complex item is then given by

$$E(T^n) = \int_0^\infty t^n \sum_{i=1}^{M} (b_i t^{b_i-1}/a_i) \exp\left(-\sum_{i=1}^{M} t^{b_i}/a_i\right) dt$$

which can be integrated by parts to yield

$$(13).\quad E(T^n) = \int_0^\infty n t^{n-1} \exp\left(-\sum_{i=1}^{M} t^{b_i}/a_i\right) dt$$

Again the mean and variance will be denoted by λ and σ^2 respectively and are determined by equation (13) to be

$$(14).\quad \lambda = \int_0^\infty \exp\left(-\sum_{i=1}^{M} t^{b_i}/a_i\right) dt$$

$$(15).\quad \sigma^2 = \int_0^\infty 2t \exp\left(-\sum_{i=1}^{M} t^{b_i}/a_i\right) dt - \lambda^2$$

Equations (14) and (15) can be evaluated by numerical methods.

If the Weibull family of probability distributions is assumed to characterize the time to failure of an item or component consisting of more than one item where the various members of the family differ only by the values of the parameters occuring in the functional form of the distribution, it is clear that the reliability, hazard, density, and cumulative distri-

bution functions, along with the various moments are functions of the parameters, a and b. Consequently, useful probability statements about these items can only be made to the extent that estimates of these parameters may be given. At the present time, point and confidence interval estimation of parameters are most frequently employed.

The maximum likelihood estimates of the parameters, a and b, will be derived for each of the sampling procedures presented in chapters III and IV, although confidence interval estimates will only be enumerated for the scale parameter, a, when the shape parameter, b, is assumed to be known in one of the sampling plans. A thorough search of recent technical publications indicates that confidence interval estimates, when both parameters are unknown, have not been derived and the derivation of these confidence interval estimates is beyond the scope of this thesis.

CHAPTER III

SAMPLING PLANS

In this chapter, several methods of obtaining experimental data to determine the maximum likelihood estimates (M.L.E.) of the parameters that appear in the functional form of the two parameter Weibull probability distribution will be discussed. Experimental data can be obtained in a variety of ways, and each particular method of obtaining experimental measurements will be defined as a sampling plan. For each of the sampling plans presented, it is assumed that the random variable T characterizes the time to failure of an item and the density function, f(t), defined by equation (3), specifies the probability law of T. It is further assumed that it is possible to make repeated measurements, denoted by $T_1, T_2, \ldots, T_n$, of the random variable T, and the measurements, $T_1, \ldots, T_n$, are independent identically distributed random variables, each with the same distribution as the random variable T. Accordingly, after selecting a random sample and performing an experiment, the observed values of the random variable T are designated by $t_1, t_2, \ldots, t_n$.

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In the case of a complex item composed of M individual items, a random sample of size n_i will be taken from the item of the ith type, for $i = 1, 2, \ldots, M$. Again the random variables T_{ij}, for fixed i and $j = 1, 2, \ldots, n_i$, are independent identically distributed random variables. Also after performing an experiment, the observed values of the random variable T_i will be denoted by $t_{i1}, t_{i2}, \ldots, t_{in_i}$. Each of the sampling plans presented are based upon random samples as defined in the above paragraph and are restatements of the sampling plans contained in reference [5] .

The method of maximum likelihood is utilized to obtain estimates of the parameters, because this procedure generally tends to have desirable properties not always shared by other methods. In brief, the method of maximum likelihood estimation consists of determining the values of the parameters that maximize the likelihood function, denoted by L. The likelihood function, L, is a function of the parameters in the joint density with fixed observations.

This maximizing process is generally accompolished by determining the values of the parameters that maximize the natural logarithm of L, since ln L is an increasing function of L. The functional form of L will vary in general with each sampling plan. Lloyd and Lipow [6] and Mood [7] provide a detailed discussion of the maximum likelihood method.

The maximum likelihood estimates of the parameters, a and b, will be denoted by $\hat{a}$ and $\hat{b}$ respectively in each of the sampling plans discussed in this paragraph. These sampling plans are outlined as follows:

SAMPLING PLAN I - Fixed sample size.

Single item case - A random sample of size n is selected and each member of the sample is tested until it fails. The total time to failure of each member of the sample is observed and recorded. If the observed failure times of the random variable T are $t_1, t_2, \ldots, t_n$, then the likelihood function is given by

$$L = \prod_{i=1}^{n} f(t_i)$$

Upon substituting equation (3) for $f(t_i)$ in the above relation and taking the natural logarithm

of L, we have

$$\ln L = n \ln b - n \ln a + \sum_{i=1}^{n} \ln t_i^{b-1} - \sum_{i=1}^{n} t_i^{b}/a$$

Therefore, if both the parameters, a and b, are unknown, the M.L.E. of these parameters are given by

$$(16).\quad \hat{a} = \sum_{i=1}^{n} t_i^{\hat{b}}/n$$

$$(17).\quad \hat{b} = n\hat{a}/\left(\sum_{i=1}^{n} t_i^{\hat{b}} \ln t_i - \hat{a} \sum_{i=1}^{n} \ln t_i \right)$$

respectively. Lloyd and Lipow [6] outline a process of iteration which will solve equations (16) and (17) for $\hat{a}$ and $\hat{b}$. In the event that the parameter, b, is assumed to be known, the M.L.E. of the parameter, a, under this sampling plan is given by

$$(18).\quad \hat{a} = \sum_{i=1}^{n} t_i^{b}/n$$

In this situation, it is also observed that a confidence interval estimate for the scale parameter can easily be determined using the statistic,

$$\left(2 \sum_{i=1}^{n} t_i^{b}\right)/a$$

which has a Chi-square distribution with 2n degrees of freedom. Complete details of confidence interval estimates can be found in Mood [7].

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Complex item case - Assuming that the complex item is composed of M individual items, the procedure is to select a random sample of size n_i from the item of the ith type, for $i = 1, 2, \ldots, M$. Each member of the ith random sample is tested until it fails, and the total time to failure is observed and recorded for each member of the ith sample. If the observed failure times of the random variable T_i are $t_{i1}, t_{i2}, \ldots, t_{in_i}$, then the likelihood function for the item of the ith type is given by

$$L_i = \prod_{j=1}^{n_i} f(t_{ij})$$

for $i = 1, 2, \ldots, M$. Therefore, upon substituting equation (3) for $f(t_{ij})$ in the above relation, we obtain

$$L_i = \prod_{j=1}^{n_i} (b_i \, t_{ij}{}^{b_i-1})/a_i \exp(-t_{ij}{}^{b_i}/a_i)$$

Accordingly, the M.L.E. of the scale and shape parameters for the item of the ith type are determined, in the same manner as for a single item, to be

$$(19). \quad \hat{a}_i = \sum_{j=1}^{n_i} t_{ij}{}^{\hat{b}_i}/n_i$$

and

$$(20). \quad \hat{b}_i = n_i \hat{a}_i / \left(\sum_{j=1}^{n_i} t_{ij}{}^{\hat{b}_i} \ln t_{ij} - \hat{a}_i \sum_{j=1}^{n_i} \ln t_{ij} \right)$$

respectively, for $i = 1, 2, \ldots, M$. Again, a process of iteration can be employed to solve for the mamimum likelihood estimates. In the situation where all of the shape parameters are assumed to be known, the M.L.E. of a_i under this sampling plan is given by

$$\hat{a}_i = \sum_{j=1}^{n_i} t_{ij}^{b_i}/n_i$$

for $i = 1, 2, \ldots, M$.

<u>SAMPLING PLAN II</u> - Fixed sample size with item truncation.

<u>Single item case</u> - A random sample of size n is placed on test simultaneously, and the life test is terminated when r members of the sample fail. The number of failures r is fixed in advance of the experiment, and r is an integer greater than zero but less than or equal to n. Using this sampling plan, the first r failures times of the random variable T are observed and recorded as the failures occur. Then, if $0 < t_1 \leq t_2 \leq \ldots, \leq t_r < \infty$ are the first r failures, Halperin [8] indicates that the likelihood function is given by

$$(21). \quad L = n!/(n-r)! \prod_{i=1}^{r} f(t_i) \left[1-F(t_r)\right]^{n-r}$$

where L is merely the density function of the first r order statistics. After substituting F(t) and f(t) of equation (3) and (4) in the above equation, the M.L.E. of the parameters, a and b, are determined in the usual manner to be given by

$$(22). \quad \hat{a} = \left[\sum_{i=1}^{r} t_i^{\hat{b}} + (n-r) t_r^{\hat{b}} \right] / r$$

$$(23). \quad \hat{b} = r\,\hat{a} / \left[\sum_{i=1}^{r} t_i^{\hat{b}} \ln t_i + (n-r) t_r^{\hat{b}} \ln t_r - \hat{a} \sum_{i=1}^{r} t_i \right]$$

Again, the values of $\hat{a}$ and $\hat{b}$ which solve equations (22) and (23) can be determined by an iteration process. In the event that the shape parameter is assumed to be known, the M.L.E. of the parameter, a, under this sampling plan is given by

$$(24). \quad \hat{a} = \left(\sum_{i=1}^{r} t_i^{b} + (n-r) t_r^{b} \right) / r$$

<u>Complex item case</u> - Assuming that the complex item is composed of M individual items, the procedure is to select a random sample of size n_i from the item of the ith type, for i = 1, 2, . . . , M. The n_i members of the ith random sample are placed on test

[illegible]

simultaneously and the ith life test is terminated when r_i members of the ith sample fail. As before, the number of failures r_i is fixed in advance of each experiment, and r_i is an integer greater than zero but less than or equal to n_i. The first r_i failures times of the random variable T_i are observed and recorded as the failures occur. If the first r_i failure times of the random variable T_i are given by $0 < t_{i1} \leq t_{i2} \leq \ldots \leq t_{ir_i} < \infty$, then the likelihood function for the item of the ith type is

$$L_i = n_i!/(n_i - r_i)! \prod_{j=1}^{n_i} f(t_{ij}) \left[1-F(t_{ir_i})\right]^{n_i - r_i}$$

for $i = 1, 2, \ldots, M$. Proceeding in exactly the same manner as with a single item, the M.L.E. of the parameters, a_i and b_i, of the item of the ith type are

$$(25). \quad \hat{a}_i = \left[\sum_{j=1}^{n_i} t_{ij}^{\hat{b}_i} + (n_i - r_i) t_{ir_i}^{\hat{b}_i} \right] / r_i$$

and

$$(26). \quad \hat{b}_i = r_i \hat{a}_i / \left[\sum_{j=1}^{n_i} t_{ij}^{\hat{b}_i} \ln t_{ij} + (n_i - r_i) t_{ir_i}^{\hat{b}_i} \ln t_{ir_i} - \hat{a}_i \sum_{j=1}^{n_i} \ln t_{ij} \right]$$

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As before, equations (25) and (26) can be solved by an iteration process for $\hat{a}_i$ and $\hat{b}_i$, for $i = 1, 2, \ldots M$. If all of the shape parameters are assumed to be known, the M.L.E. of the parameter a_i is

$$(27).\quad \hat{a}_i = \left[\sum_{j=1}^{r_i} t_{ij}^{\,b_i} + (n_i - r_i)\, t_{ir_i}^{\,b_i} \right] / r_i$$

for $i = 1, 2, \ldots, M$.

<u>SAMPLING PLAN III</u> - Fixed sample size with time truncation.

<u>Single item case</u> - A random sample of size n is selected and each member of the sample is tested for a specified time, say t_o. The time t_o, where t_o is greater than zero but less than infinity, is defined as the truncation time. The number of failures denoted by R that occur in the time interval, $(0, t_o)$, are observed and the R failure times of the random variable T recorded. Using this sampling procedure, R is of course a random variable which may take the value r, where $r = 0, 1, 2, \ldots n$. If the observed failure times are $t_1, t_2, \ldots, t_r$, then the likelihood function is

$$L = \binom{n}{r} \prod_{i=1}^{r} f(t_i) \left[1 - F(t_o) \right]^{n-r} \quad \text{for } t_i < t_o.$$

The likelihood function for this sampling plan is formally derived in Appendix A. Again, substituting

$F(t)$ and $f(t)$ of equations (3) and (4) in the above relation, the M.L.E. of a and b are derived in the usual manner and determined to be given by

$$(28).\quad \hat{a} = \left[\sum_{i=1}^{r} t_i^{\hat{b}} + (n-r)t_o^{\hat{b}} \right] / r$$

$$(29).\quad \hat{b} = r\,\hat{a} \Big/ \left[\sum_{i=1}^{r} t_i^{\hat{b}} \ln t_i + (n-r)t_o^{\hat{b}} \ln t_o - \hat{a} \sum_{i=1}^{r} t_i \right]$$

respectively. As before, an iteration process can be employed to solve equations (28) and (29) for $\hat{a}$ and $\hat{b}$. Also, under this sampling plan the M.L.E. of the scale parameter is given by

$$(30).\quad \hat{a} = \left[\sum_{i=1}^{r} t_i^{b} + (n-r)\, t_o^{b} \right] / r$$

when the shape parameter is assumed to be known.

<u>Complex item case</u> - If the complex item is assumed to be composed of M individual items, the experimental procedure is to select a random sample of size n_i from the item of the ith type, for $i = 1, 2, \ldots, M$. The sample of the ith type is tested for a specified truncation time, t_{i_o}, and the number of failures denoted by R_i that occur in the time interval, $(0, t_{i_o})$, are ob-

served and the R_i failure times of the random variable T_i recorded. As before, the number of failures, R_i, is a random variable which may take the value r_i, where $r_i = 1, 2, \ldots, n_i$. If $t_{i1}, t_{i2}, \ldots, t_{ir_i}$ are the observed values of the r_i failure times; then using the likelihood function,

$$L_i = \binom{n_i}{r_i} \prod_{j=1}^{r_i} f(t_{ij}) \left[1-F(t_{i_o})\right]^{n_i-r_i},$$

the M.L.E. of a_i and b_i are determined to be given by

$$(31). \quad \hat{a}_i = \left[\sum_{j=1}^{r_i} t_{ij}^{\hat{b}_i} + (n_i-r_i)t_{i_o}^{\hat{b}_i} \right] / r_i$$

$$(32). \quad \hat{b}_i = r_i\hat{a}_i / \left[\sum_{j=1}^{r_i} t_{ij}^{\hat{b}_i} \ln t_{ij} + (n_i-r_i)t_{i_o}^{\hat{b}_i} \ln t_{i_o} - \hat{a}_i \sum_{j=1}^{r_i} \ln t_{ij} \right]$$

for the item of the ith type, where $i = 1, 2, \ldots M$. If all the shape parameters are assumed to be known, the M.L.E. of the parameter a_i is

$$(33). \quad \hat{a}_i = \left[\sum_{j=1}^{r_i} t_{ij}^{b_i} + (n_i-r_i)t_{i_o}^{b_i} \right] / r_i$$

for $i = 1, 2, \ldots, M$.

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Thus, it is readily seen that considerable calculations are required to solve for $\hat{a}$ and $\hat{b}$ in each of the above sampling plans. But by utilizing a high speed electronic computer, the iteration process required to solve for $\hat{a}$ and $\hat{b}$ is readily accompolished. Kao [9] outlines a graphical method of obtaining a first approximation for $\hat{a}$ and $\hat{b}$. If these first approximations of $\hat{a}$ and $\hat{b}$ are provided to the computer as the initial trial, the iteration method should quickly converge to a solution for $\hat{a}$ and $\hat{b}$. In the next chapter, the above sampling plans, assuming the parameters, a and b, are unknown, will be utilized in the proposed models for analyzing Polaris Missile System Trouble Failure Reports. It will be further assumed that high speed electronic computers are available to accompolish the calculations required to solve for $\hat{a}$ and $\hat{b}$.

CHAPTER IV

APPLICATION TO POLARIS MISSILE SYSTEM

In this chapter, three models are proposed which utilize Polaris Missile System (PMS) Trouble Failure Report (TFR) as experimental data for determining M.L.E. of parameters. By following the procedures proposed in the models below, it may be possible to obtain a useful point estimate of quality indices, such as mean time to failure and reliability, of certain items composing the missile system. The models are applicable primarily to missile items for which a life history is maintained. An item will be considered to have a life history if:

(1). It is possible to distinguish each item by some identification system from all other items of the same type.

(2). A permanent record is maintained of the total operating time to failure.

(3). Assuming that the item does not fail, it is possible to estimate accurately the total operating time accumulated by the item at some future date.

In each of the models, the total operating time to failure of an item is assumed to be a random variable

T that is specified by the two parameter Weibull probability distribution with both parameters considered to be unknown. Accordingly, if n items of the same type are selected at random, this sample is assumed to consist of n independent identically distributed random variables which are characterized by the same probability law as T. It is further assumed that none of the items have accumulated operating time prior to their purchase by the United States Navy. Also, an item, which has failed and been repaired, is considered to be the same as a newly purchased item with no accumulated operating time. In each of the models, the operating time of the items is accumulated by check outs or other reasons for operating the item, and the TFR which provides the total operating time to failure is used as the observed value of the random variable. Naturally, it is assumed that the time to failure reported in a TFR is accurate.

The first model proposed is Sampling Plan I which is discussed in Chapter III. Consequently, for a sample of size n, the M.L.E. of a and b are given by equations (16) and (17), respectively, and a point estimate for the reliability and mean

time to failure can be obtained by substituting the calculated values of $\hat{a}$ and $\hat{b}$ into equations (5) and (7), respectively. In the case of a complex item, the procedure is the same, except that the values are obtained for the M.L.E. of a_i and b_i for $i = 1, 2, \ldots, M$, with equations (19) and (20). Again, a point estimate of the reliability and mean time to failure can be obtained by substituting the values of $\hat{a}_i$ and $\hat{b}_i$ into equations (11) and (14), respectively. Notice that equation (14) must be evaluated by numerical methods to obtain a point estimate of λ. The limitation of this model is the waiting time required to observe the nth failure.

Prior to outlining the next two models which are applications of Sampling Plans II and III, it is necessary to present additional background concerning the PMS. Consider an item of a given type, say X, which continues to accumulate operating time by check outs until it fails. Because the PMS is an expanding program, items of type X are constantly being added to the system and also accumulating operating time. Consequently, if n items of type X are selected as a sample, each of the items could

have accumulated varying amounts of operating time depending upon when the item was purchased by the Navy. The effect of the above situation is to complicate item and time truncation sampling plans as discussed in Chapter III. But Sampling Plans II and III can still be used to obtain M.L.E. of the parameters, if a life history is kept for items of type X, as follows.

The second method for obtaining point estimates of quality indices employs Sampling Plan II. The procedure is to select P samples of size n_j, where $j = 1, 2, \ldots, P$, from items of type X. The items in the jth sample are selected so that each of the members of this sample are expected to have accumulated approximately the same amount of operating time at a future date. For each of the P samples, fix in advance the number of observed failures, denoted by r_j, which will terminate the jth experiment. If equation (21) is used as the likelihood function for the jth sample of size n_j and if the P samples are assumed to be independent, the likelihood function, denoted by $\mathcal{L}$, of the P samples is given by

$$\mathcal{L} = \prod_{j=1}^{P} n_j!/(n_j-r_j) \prod_{i=1}^{r_j} f(t_i) \left[1-F(t_{r_j})\right]^{n_j-r_j}$$

Upon substituting F(t) and f(t) of equations (3) and (4) in the above relation, the M.L.E. of the parameters, a and b, are determined in the usual manner to be given by

$$(34).\quad \hat{a} = \sum_{j=1}^{P}\left\{\left[\sum_{i=1}^{n_j} t_i^{\hat{b}} + (n_j - r_j)t_{r_j}^{\hat{b}}\right]/r_j\right\}$$

$$(35).\quad \hat{b} = \sum_{j=1}^{P}\left\{\hat{a}\, r_j/\left[\sum_{i=1}^{n_j} t_i^{\hat{b}} \ln t_i + (n_j - r_j)t_{r_j}^{\hat{b}} \ln t_{r_j} - \hat{a}\sum_{i=1}^{n_j} t_i\right]\right\}$$

Therefore, a point estimate of a quality index can be obtained by substituting equations (34) and (35) into the equation which defines the quality index desired. In the case of a complex item, the method is completely analogous except there are M items of different types. Again, life histories must be maintained for each type of item.

The third model uses Sampling Plan III. As in the second model, P samples of size n_j, where $j = 1, 2, \ldots, P$, are selected from items of type X so that each member of the jth sample is expected to have accumulated the same amount of operating time at a future date. A truncation time, denoted

by t_{j_o}, is selected for each of the j samples and the experiment is considered to be terminated after the truncation time has elasped for each of the j samples of size n_j, where $j = 1, 2, \ldots, P$. As in Sampling Plan III, the number of failures, denoted by R_j, which occur during the truncation time, t_{j_o}, in the jth sample is a random variable. Assuming that each of the P samples are independent, the likelihood function, denoted by $\mathcal{L}$, is

$$\mathcal{L} = \prod_{j=1}^{P} L_j$$

where L_j is the likelihood function of Sampling Plan III. Proceeding as in Sampling Plan III, the M.L.E. of a and b are given by

$$(36). \quad \hat{a} = \sum_{j=1}^{P} \left\{ \left[\sum_{i=1}^{n_j} t_i^{\hat{b}} + (n_j - r_j) t_{j_o}^{\hat{b}} \right] / r_j \right]$$

and

$$(37). \quad \hat{b} = \sum_{j=1}^{P} \left\{ r_j \hat{a} / \left[\sum_{i=1}^{n_j} t_i^{\hat{b}} \ln t_i + (n_j - r_j) t_{j_o}^{\hat{b}} \ln t_{j_o} - \hat{a} \sum_{i=1}^{n_j} t_i \right] \right]$$

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In the case of a complex item the procedure is completely analogous. Therefore, point estimates of quality indices can be determined in both the single and complex item case by substituting the equations for the M.L.E. of a and b into the equation for the desired quality index.

In the last two models, the selection of the members of the jth sample should occur naturally since a group of items of a given type is usually issued to a submarine at the same time. These items should accumulate operating time at the same rate through check outs until failure. The time when each of the items is issued can be determined by examining the life history of the items; consequently, the members of the jth sample can be determined for either model two or three.

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APPENDIX A

In Sampling Plan III, n items are tested until a specified time, t_o, has elasped. The number of failures denoted by R that occur in the time interval $(0,t_o)$ are observed and the R failure times recorded. Using this sampling procedure, R is a random variable which may take the value, r, where $r = 0, 1, \ldots, n$. As before, equations (3) and (4) define the probability density and cumulative distribution functions respectively of the random variable T. The discussion which follows is a formal derivative of the likelihood function for Sampling Plan III.

Maximum likelihood estimators are functions of the data obtained from an experiment. In this case, the experiment is to specify a truncation time t_o and observe the number of failures, R, and the failure times, T_i, which occur in the time interval $(0,t_o)$; consequently, the joint density function of the number of failures, R, and the times to failure, T_i, given that the failures occur in the time interval $(0,t_o)$ would be appropriate for determining the likelihood function.

Let A be defined as the event,

$(T_1 < t_1, T_2 < t_2, \ldots, T_R < t_R)$ where the T_i are not ordered and $t_i < t_o$ for $i = 1, 2, \ldots, R$.

Let B be defined as the event, $(R = r)$, where $r = 0, 1, \ldots, n$.

Let C be defined as the event that the failures which occur do so before time t_o.

Let the event, AB, be expressed as $(T_1 < t_o, \ldots, T_r < t_o)$. Then the problem is to obtain the distribution function which represents the probability statement $P(AB|C)$. Using conditional probability relations, it is observed that

$$P(AB|C) = \frac{P(ABC)}{P(BC)} \cdot \frac{P(BC)}{P(C)} = P(A|BC)P(B|C)$$

where

$$P(A|BC) = P(T_1 < t_1, \ldots, T_R < t_R | B \cap C)$$

$$= \frac{\binom{n}{r} P\left[(T_1 < t_1, \ldots, T_r < t_r, T_{r+1} > t_o, \ldots, T_n > t_o) \cap B \cap C\right]}{P(B \cap C)}$$

$$= \frac{\binom{n}{r} \prod_{i=1}^{r} F(t_i) \left[1 - F(t_o)\right]^{n-r}}{\binom{n}{r} \left[F(t_o)\right]^r \left[1 - F(t_o)\right]^{n-r}} \quad \text{for } t_i < t_o$$

$$= \frac{\prod_{i=1}^{r} F(t_i)}{\left[F(t_o)\right]^r} \quad \text{for } t_i < t_o$$

and

$$P(B|C) = \binom{n}{r}\left[F(t_o)\right]^r \left[1-F(t_o)\right]^{n-r}$$

therefore

$$F(t_1,t_2,\ldots,t_r,R=r) = \binom{n}{r}\prod_{i=1}^{r} F(t_i) \left\{1-F(t_o)\right\}^{n-r}$$
$$\text{for } t_i < t_o$$

Accordingly, the joint density function is obtained by taking the partial derivatives of the above distribution function with respect to t_i for $i=1,\ldots,r$ and is given formally by

$$f(t_1,\ldots,t_r,R=r) = \binom{n}{r}\prod_{i=1}^{r} f(t_i) \left\{1-F(t_o)\right\}^{n-r}$$
$$\text{for } t_i < t_o$$